Role of Graphs for Multi-Agent Systems and Generalization of Euler's Formula

Qasim Ali, and Sergio Montenegro
Institute of Aerospace Information Technology, University of Würzburg, Würzburg 97074, Germany
Corresponding author: Qasim Ali (qasim.ali@uni-wuerzburg.de)

Abstract—**The historical problem of *Seven Bridges of Königsberg* caused the birth of graph theory. A number of practical problems involving networks may be appropriately represented by the graphs that facilitate problem formulation and analysis process. Communication topology for networks involving a large number of units, like multi-agent system, may be conveniently examined using the notion of graph theory. To facilitate the formulation of such problems, an appropriate mathematical solution is to represent the graph with the help of Laplacian matrix. Eigenvalues of Laplacian matrix are the main focus of this paper. Same have been exploited to give an insight into the graph / subgraph properties, and to generalize the well-known Euler's Formula in order to make it applicable for graphs as well as subgraphs. A modified Euler's formula is also presented. Effects of addition and removal of communication links for a given number of agents are examined. Effects of addition and removal of agents, and portioning a graph into subgraphs is also the focus of our present study.**

Keywords—distributed control, eigenvalues, Euler's formula, graph theory, Laplacian matrix, Multi-agent system

I. INTRODUCTION

Graph theory was first introduced by the famous mathematician Leonhard Euler in the 18^{th} century during an endeavor to solve the historical problem of *Seven Bridges of Königsberg* [1]. Since then graph theory has evolved and today we find its uses in many domains and disciplines, for e.g. to model the information exchange among units of a Multi-Agent System (MAS) and to represent the computation flow and data organization in computer networks. Graph theory has strong links with distributed / cooperative control schemes to efficiently handle communication topology affairs. It has been exploited by a number of researchers in the domain of MAS, formation and network control. Examples include the use of directed graphs for formation control [2], and Unmanned Aerial Vehicle (UAV) swarm modelling where the leader-follower scheme of aerial vehicles was extended to an arbitrary number of units [3]. The problem formulation is accommodated with the conversion of a graph into the *Laplacian matrix.* This matrix gives an insight into the communication topology, properties of the underlying graph, and graph connectivity through its eigenvalues. Laplacian matrix eigenvalues were identified as an important object of study in [4]. Eigenvalues of Laplacian matrix were exploited in [5] to determine the effects of communication topology on formation stability, and Laplacian matrix itself was used to represent the sensing of relative position of units in a formation. A class of graphs called *weakly connected graphs* were studied in [6] to represent the formation of multiple vehicles that are weakly connected. Alternatives to Laplacian-based consensus algorithms in computer science are Gossip-based algorithms such as the push-sum protocol [7].

Our analysis framework for the present work is based on tools from matrix theory and graph theory. The graphs meant for inter-agent communications for a MAS have been given due consideration. A physical interpretation of presented results may be the information flow within a formation or swarm of aerial vehicles that plays an important role towards stability. For large formations or swarms, a single leader may not efficiently control the whole formation and hence multiple leaders may be required. Handling of communication topology for large formations / swarms of aerial vehicles is facilitated with the use of graph theory.

Communication links between the agents may be directional or bidirectional, depending upon the course of information flow. Directed and undirected graphs correspond to directional and bidirectional communication links respectively. Addition and removal of communication links between the agents affects the Laplacian matrix, which will also be considered in this paper. Subgraphs are also the focus of our study, where these have been related to multiple formations or clusters of agents represented by a single Laplacian matrix. Some results for the properties of Laplacian matrix and associated eigenvalues are also discussed. There are some interesting properties related to the second smallest eigenvalue of Laplacian matrix, known as *Algebraic Connectivity*. Work has been done to increase the algebraic connectivity in [8]. Reference [9] suggests to choose a set of edges effectively in order to maximize the algebraic connectivity for a given number of vertices. Further details on algebraic connectivity are provided in section II.

A lot of work has been done in the domain of *Euler Characteristic* for 3D objects, e.g. [10]. The main contribution of this paper is the generalization of Euler's formula that holds true for the planar graphs as well as subgraphs. An endeavor has been made to relate the Euler's formula to the eigenvalues of underlying Laplacian matrix. For our present study, we focus on directed and undirected graphs and related Laplacian matrices. However, normalized Laplacian matrix is not the focus of our present paper.

Further description of this paper is organized as follows. Section II describes preliminary background on graph theory. Section III provides definitions and propositions in the domain

This work was sponsored by Higher Education Commission (HEC), Pakistan and Deutscher Akademischer Austausch Dienst (DAAD), Germany.

of graph theory and generalization of Euler's formula. Finally conclusions are drawn in section IV.

II. PRELIMINARY BACKGROUND

A *graph* G with a set of vertices V and edges E is generally represented as $G = (V, E)$. The number of vertices (or nodes) and edges will be represented as $|V|$ and $|E|$ respectively. *Well-Connectedness* of a graph may be defined as the number of edges in a graph. *Fully connected graph*, also called a *complete graph*, is one where all possible connections between the agents exist. *Degree of a graph* means maximum number of edges for any of its vertices [7]. Graphs are broadly classified as *directed graphs* and *undirected graphs*.

If all vertices of a graph are connected then there is only one eigenvalue zero (simple eigenvalue) of related Laplacian matrix. However, when a graph is partitioned into multiple *subgraphs* then algebraic multiplicity of zero eigenvalue represents the number of subgraphs. Individual connected components of a graph are also called as s*ubgraphs*. For a node to be a part of a graph or subgraph, it must have at least one edge connected to it. In physical terms, an agent (a vertex) must communicate with at least one other agent to be part of a graph or subgraph. Edge denotes a communication link.

A *loop* may be defined as a closed path that does not enclose another closed path. For communication among a large number of units (e.g. a swarm), a loop indicates redundant information flow from/to multiple units. It also gives an indication for connectedness of a graph. Note that a *loop* here does not mean a *self-loop* that corresponds to connection of a vertex with itself. A loop is also referred to as a *cycle* in graph theory. *Tree* in graph theory means *no loops*.

Now we define some matrices in the domain of graph theory. *Degree matrix D* is a *n* x *n* diagonal matrix that gives the information about the number of edges (degree) of each vertex. Degree of a vertex is given by the number of its neighbors. If elements of a degree matrix are represented as $D[i,j]$ where i and j represent the vertices of a matrix then:

$$D[i,j] \;:=\; \begin{cases} \deg(v_i) & if\ i = j \\ 0 & otherwise \end{cases}$$

Here $\deg(v_i)$ denotes degree of vertex *i*. An example of a degree matrix is given below that corresponds to an arbitrary graph of Fig. 1.

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Adjacency matrix A is a square *n* x *n* matrix that gives information about adjacency of vertices in an undirected graph. Adjacency matrix is defined as:

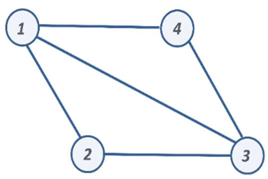


Fig. 1. An undirected graph with four vertices

$$A[i,j] \;:=\; \begin{cases} 1 & if\ an\ edge\ exists\ between\ vertex\ i\ and\ j \\ 0 & if\ no\ edge\ exists\ between\ vertex\ i\ and\ j \end{cases}$$

An example of adjacency matrix, corresponding to the graph in Fig. 1, is as under:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

The *Laplacian matrix* $\mathcal{L}$ is a square *n* x *n* matrix that may be defined, using the notion of degree matrix *D* and adjacency matrix *A*, as under:

$$\mathcal{L} = D - A$$

Laplacian matrix may also be directly defined as follows:

$$\mathcal{L}[i,j] \;:=\; \begin{cases} \deg(v_i) & if\ vertex\ i = j \\ -1 & if\ i \neq j\ and\ i\ \&\ j\ are\ adjacent \\ 0 & otherwise \end{cases}$$

$\deg(v_i)$ is defined as before. Laplacian matrix for the graph in Fig. 1 is given below:

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Laplacian matrix $\mathcal{L}$ being a symmetric and positive semi-definite matrix has special properties. Its lowest eigenvalue is always zero and all other eigenvalues are positive and real. $\boldsymbol{\mathcal{L}}$ has a right eigenvector of ***1*** associated with the zero eigenvalue because of the identity $\boldsymbol{\mathcal{L}1}$=0, where ***1*** is a vector of all ones. This implies that the row sum of all elements in a $\boldsymbol{\mathcal{L}}$ matrix is zero. Laplacian matrix is only diagonalizable for undirected topologies. Laplacian matrix (for undirected graph) is always singular, as zero is always its eigenvalue. Laplacian matrix may indicate the type of control scheme being used; e.g. Laplacian matrix is an Identity matrix for *decentralized control* [11].

Incidence matrix Inc is designed for directed graphs which is useful for scenarios like current flow in electric circuits, and information flow among different units in a MAS etc. The number of vertices and edges of a graph correspond to number of columns and rows of an incidence matrix respectively. An arbitrary directed graph is shown in Fig. 2. Corresponding Incidence matrix is as under:

$$Inc = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Note that edges are also required to be numbered (in addition to vertices) for writing an incidence matrix. Also note that loop corresponds to linearly dependent rows. For e.g.,

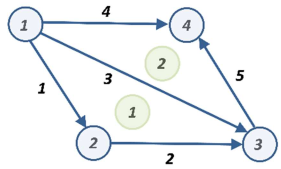


Fig. 2. A directed graph with four vertices

row 3 is a sum of rows 1 and 2; similarly row 4 is a sum of rows 3 and 5 of incidence matrix. Two loops are shown in light green color in Fig. 2. Unlike adjacency matrix, degree matrix and Laplacian matrix; an incidence matrix in not symmetric. Incidence matrix and Graph Laplacian matrix are interlinked as $\mathcal{L} = Inc * Inc^T$ [9], where the superscript ^{T} shows matrix transpose.

Some of the agents in a swarm may have superior sensing, computation or communication abilities, called as leaders [12]. Information flow is usually from leader to the followers. For Fig. 2, we may consider vertex 1 as a leader (that only provides information to other units) and the remaining vertices as the followers.

Every agent in a formation only needs to know the states of its neighboring agent, not all in the formation. *Neighborhood* of the units is defined in terms of metric distance or topological distance. Neighborhood based on metric rules may be defined as:

Definition 1: A node j is a neighbor of node i if l_2 norm of displacement vector d between the two nodes is less than δ. Mathematically:

$$\|d\|_2 < \delta \tag{1}$$

where δ is a bounded number that depends on the range of communication media. On the other hand, *topological distance model* relies on surrounding units which are immediate neighbors irrespective of their distances**.** Interactions are thus based on topological rules rather than metric rules.

A. Algebraic Connectivity and Its Utility

Spectrum of a Laplacian matrix $\mathcal{L}$ is ordered as $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots . \lambda_n\,(G)$. The zero eigenvalue is known as the trivial eigenvalue of $\mathcal{L}$. The second smallest eigenvalue of Laplacian matrix is called the *algebraic connectivity* of graph G, and the corresponding normalized eigenvector is called the *Fiedler Vector* [9]. The second smallest eigenvalue $\lambda_2 > 0$ is a necessary and sufficient condition for connectivity of graph G. If it is also zero then it implies that there exist two subgraphs. Algebraic multiplicity of zero eigenvalue represents the number of subgraphs.

Algebraic connectivity is considered to be a measure of how well connected a graph is [9]. This value monotonically increases when we increase the number of edges in the graph [13]. It suggests that algebraic connectivity either maintains its value or it increases but never decreases with the increase in number of edges of the graph. Note that algebraic connectivity is not an absolute measure rather a relative measure of graph connectivity. It implies that the second smallest eigenvalue may not always give an absolute measure of connectedness of a graph, as two graphs having different number of edges may have the same algebraic connectivity. Please see rows 4 – 7 of Table I. Reference [9] states that if the second eigenvalue has an algebraic multiplicity *r*, then we need to add *r* number of edges before the second smallest eigenvalue increases from its current value. In other words, the second smallest eigenvalue along with its algebraic multiplicity gives complete information about the connectedness of a graph. The number of vertices in a graph corresponds to an upper limit on algebraic connectivity. The maximum attainable value of algebraic connectivity may not exceed the number of vertices in the graph [13].

Algebraic connectivity is also regarded as a measure of stability and robustness of the networked dynamic systems [8]. It is also used in analyzing the synchronization of coupled dynamical systems [14], and is a measure of performance and speed of convergence of consensus algorithms [7].

III. Graph Properties And Propositions

Tables I and II show some arbitrary graphs and corresponding Laplacian matrices and eigenvalues as a ready reference to provide an insight into forthcoming discussion in this section. Table I also indicates that it does not matter how we number the vertices, eigenvalues of resultant Laplacian matrix remain unchanged.

For an Incidence matrix Inc, the following relations hold [15]:

$$dim\, N(Inc^T) = |E| - |V| + 1 \tag{2}$$

or,

$$\dim N(Inc^T) = |L| \tag{3}$$

Here $dim\, N$ represents dimensions of null space, and $|L|$ the number of loops. Equations (2) and (3) are only valid for directed graphs.

Proposition 1: Let ${m_1}^L$ represent the algebraic multiplicity of zero eigenvalues of associated Laplacian matrix, then following relation holds for a directed graph:

$$|L| = {m_1}^L \tag{4}$$

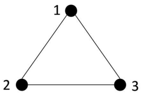
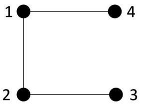
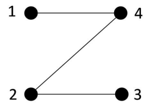
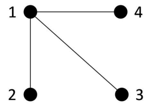
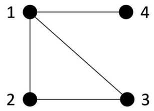
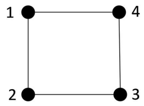
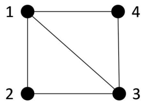
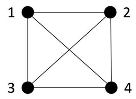
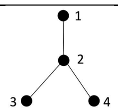
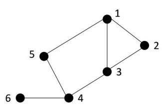
Though a relation between product of nonzero eigenvalues of Laplacian matrix and number of spanning trees of a connected graph already exists in the form of *matrix tree theorem* [16], we exploit this measure as follows:

Definition 2: Product (or trace) of all nonzero eigenvalues of a Laplacian matrix $\mathcal{L}$ is a measure for nodes connectivity of a given graph (or subgraph), as it always increases with the increase in number of edges. The product (or trace) may also be considered as a measure of "feasibility of information exchange".

Definition 3: For a Laplacian matrix of an undirected graph, sum of its eigenvalues is equal to number of communication paths. Note that if agent 1 sends information to agent 2 and vice versa, these are taken as two communication paths. Mathematically it may be written as:

$$\frac{\sum \lambda}{2} = |E| \tag{5}$$

TABLE I. SAMPLE GRAPHS AND CORRESPONDING METADATA

Topology	Laplacian Matrix	Spectrum
	$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$	λ = 0, 3, 3 ∑ λ = 6 *∏ λ = 9
	$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$	λ = 0, 0.59, 2, 3.41 ∑ λ = 6 *∏ λ = 4
	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$	λ = 0, 0.59, 2, 3.41 ∑ λ = 6 *∏ λ = 4
	$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$	λ = 0, 1, 1, 4 ∑ λ = 6 *∏ λ = 4
	$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$	λ = 0, 1, 3, 4 ∑ λ = 8 *∏ λ = 12
	$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$	λ = 0, 2, 2, 4 ∑ λ = 8 *∏ λ = 16
	$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$	λ = 0, 2, 4, 4 ∑ λ = 10 *∏ λ = 32
	$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$	λ = 0, 4, 4, 4 ∑ λ = 12 *∏ λ = 64
	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$	λ = 0, 1, 1, 4 ∑ λ = 6 *∏ λ = 4
	$\begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$	λ = 0, 0.72, 1.68, 3, 3.70, 4.89 ∑ λ = 14 *∏ λ = 66

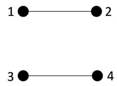
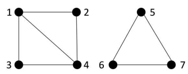
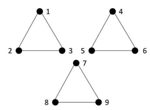
* Product of nonzero eigenvalues

$\sum \lambda$ denotes the sum of eigenvalues of Laplacian matrix. Equation (5) is applicable to graphs as well as subgraphs. As trace of a matrix (whether diagnolized or otherwise) is equal to sum of its eigenvalues, the trace of a Laplacian matrix shows total number of connections. Same is also evident from Tables I and II.

Definition 4: If a graph is fully connected (all possible connections in the graph exist), then following holds true:

- All elements in the Laplacian matrix are nonzero.
- All nonzero eigenvalues of Laplacian matrix are same.
- Algebraic multiplicity of nonzero eigenvalues indicates the number of edges that each vertex has with other vertices.

TABLE II. SAMPLE SUBGRAPHS AND CORRESPONDING METADATA

Topology	Laplacian Matrix	Spectrum
	$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$	λ = 0, 0, 2, 2 ∑ λ = 4 *∏ λ = 4
	$\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$	λ = 0, 0, 2, 3, 3, 4, 4 ∑ λ = 16 *∏ λ = 288
	$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$	λ = 0, 0, 0, 3, 3, 3, 3, 3, 3, ∑ λ = 18 *∏ λ = 729

* Product of nonzero eigenvalues

- Numeric value of nonzero eigenvalues shows the total number of vertices in the graph.

Corollary 1: Half of the number of zeros in a Laplacian matrix inform about the number of edges that need to be added to make the graph fully connected.

Definition 4 and Corollary 1 are useful for small graphs. For large formations, fully connected graphs may not be desirable. Table III shows two subgraphs and their merging into a single graph in two different ways. Corresponding changes in Laplacian matrices and eigenvalues are also depicted in the table.

A. Generalization of Euler's Formula for Planar Graphs

The relation between Euler's formula and eigenvalues of underlying Laplacian matrix has been determined. Euler's formula for a planar graph is as follows [15]:

$$|V| - |E| + |L| = 1 \tag{6}$$

This formula is applicable to the connected graphs (directed and undirected). For undirected graphs, algebraic multiplicity of zero eigenvalue of corresponding Laplacian matrix is one.

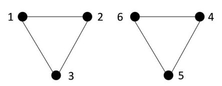
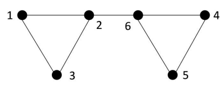
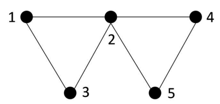
Proposition 2: For graphs as well as subgraphs (when represented as a single Laplacian matrix), Euler's formula may be generalized as:

$$|V| - |E| + |L| = 1 * {m_1}^L \tag{7}$$

${m_1}^L$ is as defined before. Equation (7) is a generalization of Euler's formula, which is applicable not only for a connected graph but also multiple subgraphs; whereas (6) is only applicable for one connected graph. Remember that for (7) to hold good, only one Laplacian matrix is to be exploited for representation of multiple subgraphs. This is useful when, for example, multiple formations of aerial vehicles are to be controlled through a single ground station (or some other platform) using distributed / cooperative control schemes.

Equation (7) may be readily verified in the light of data given in all the tables. Table III is also of interest, where two subgraphs merge to one graph and algebraic multiplicity and other parameters also change accordingly.

TABLE III. MERGING OF SUBGRAPHS

Topology	Laplacian Matrix	Spectrum
	$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$	λ = 0, 0, 3, 3, 3, 3 $\sum\lambda = 12$ $*\prod\lambda = 81$
	$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{bmatrix}$	λ = 0, 0.44, 3, 3, 3, 4.56 $\sum\lambda = 14$ $*\prod\lambda = 54$
	$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}$	λ = 0, 1, 3, 3, 5 $\sum\lambda = 12$ $*\prod\lambda = 45$

* Product of nonzero eigenvalues

As we know that dimensions of null space correspond to the algebraic multiplicity of eigenvalue zero, so from (7) it follows that:

$$|V| - |E| + |L| = \dim N(L) \tag{8}$$

B. Euler's Modified Formula

Keeping in view previous discussion, Euler's formula may be written in modified form as:

$$|V| - \frac{\sum\lambda}{2} + |L| = 1 \tag{9}$$

or,

$$\sum\lambda = 2 * (|V| + |L| - 1) \tag{10}$$

Equation (10) is applicable for a connected graph. It may be generalized as:

$$\sum\lambda = 2 * (|V| + |L| - {m_1}^L) \tag{11}$$

Equation (11) is applicable for multiple subgraphs as well as for a single connected graph. As sum of eigenvalues is the same as trace of a square matrix, so (10) may be formulated as:

$$Trace = 2 * (|V| + |L| - 1) \tag{12}$$

C. Generalization of Euler's Formula for 3D Graphs

For three dimensional connected graphs, Euler's formula is as follows:

$$|V| - |E| + |F| = 2 \tag{13}$$

Here $|F|$ represents the number of faces. Now generalizing it for multiple subgraphs:

$$|V| - |E| + |F| = 2 * {m_1}^L \tag{14}$$

This formula is applicable for multiple geometrical shapes when they are represented as a single Laplacian matrix. Now (13) may be reformulated using (5) as follows:

$$\sum\lambda = 2 * (|V| + |F| - 2) \tag{15}$$

that is applicable for single graph only. It may be generalized to make it applicable for multiple connected parts as well as for one single graph, as follows:

$$\sum\lambda = 2 * (|V| + |F| - 2 * {m_1}^L) \tag{16}$$

IV. CONCLUSION AND FUTURE WORKS

Euler's formula has been generalized to make it applicable to multiple subgraphs while exploiting the eigenvalues of underlying Laplacian matrix. Application of presented results may be found in swarm of aerial vehicles, formation flight, MAS and Kirchoff's current law etc. These are envisaged to be helpful for analysis of the network of networks. Network dynamics and control for large number of agents may be conveniently investigated using the notion of graph theory.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers and the editors for their time and efforts in reviewing the paper and giving valuable suggestions which greatly improved this paper. The authors would also like to thank Ms Shiloe Mokay for reviewing this paper.

REFERENCES

[1] G. Alexanderson, "About the cover: Euler and Königsberg's bridges: A historical view," *Bulletin of the American mathematical society,* vol. 43, no. 4, pp. 567-573, 2006.

[2] A. P. Popov and H. Werner, "A robust control approach to formation control," in *Proc. ECC*, Aug. 2009, pp. 4428-4433.

[3] A. Chapman and M. Mesbahi, "UAV swarms: Models and effective interfaces," in *Handbook of Unmanned Aerial Vehicles*. Dordrecht, The Netherlands: Springer, 2014, pp. 1987-2019.

[4] L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Physical Review Letters,* vol. 80, no. 10, pp. 2109, 1998.

[5] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control,* vol. 49, no. 9, pp. 1465 - 1476, Sep. 2004.

[6] R. Olfati-Saber and R. M. Murray, "Distributed cooperative control of multiple vehicle formations using structural potential functions," in *Proc. 15th IFAC World Congr.,* 2002, pp. 242-248.

[7] R. Olfati-Saber, A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE,* vol. 95, no. 1, pp. 215-233, 2007.

[8] Y. Kim and M. Mesbahi, "On maximizing the second smallest eigenvalue of a state-dependent graph Laplacian," *IEEE Trans. Autom. Control,* vol. 51, no. 1, pp. 116-120, 2006.

[9] A. Ghosh and S. Boyd, "Growing well-connected graphs," in *Proc. 45th CDC*, Dec. 2006, pp. 6605-6611.

[10] T. Leinster, "The Euler characteristic of a category," *Doc. Math,* vol. 13, pp. 21-49, 2008.

[11] S. Tonetti, M. Hehn, S. Lupashin, and R. D'Andrea, "Distributed control of antenna array with formation of UAVs," in *Proc. 18th IFAC World Congr.*, 2011, pp. 7848-7853.

[12] S. Martini, M. Egerstedt, and A. Bicchi, "Controllability decompositions of networked systems through quotient graphs," in *Proc. 47th CDC*, Dec. 2008, pp. 5244-5249.

[13] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak mathematical journal,* vol. 23, no. 2, pp. 298-305, 1973.

[14] C. W. Wu, "Synchronization in networks of nonlinear dynamical systems coupled via a directed graph," *Nonlinearity*, vol. 18, no. 3, pp. 1057-1064, Feb. 2005.

[15] G. Strang, *Introduction to Linear Algebra.* 4th ed. Wellesley, USA: Wellesley - Cambridge Press, 2009.

[16] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*. Springer Science & Business Media, 2011.